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Exhausters, optimality conditions and related problems

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Abstract The notions of exhausters were introduced in (Demyanov, Exhauster of a positively homogeneous function, Optimization 45, 13–29 (1999)). These dual tools (upper and lower exhausters) can be employed to describe optimality conditions and to find directions of steepest ascent and descent for a very wide range of nonsmooth functions. What is also important, exhausters enjoy a very good calculus (in the form of equalities). In the present paper we review the constrained and unconstrained optimality conditions in terms of exhausters, introduce necessary and sufficient conditions for the Lipschitzivity and Quasidifferentiability, and also present some new results on relationships between exhausters and other nonsmooth tools (such as the Clarke, Michel-Penot and Fréchet subdifferentials).

Keywords Positively homogeneous function · Optimality conditions · Upper and lower exhausters · Proper and adjoint exhausters · Unconstrained optimization problems · Quasidifferentiability · The Michel-Penot subdifferential · The Clarke subdifferential · The Fréchet subdifferential

1 Introduction

The main tool in the study of smooth functions is the gradient. By means of the gradient one is able, e.g., to get a first-order approximation of the function under study, to describe optimality conditions, to find steepest ascent and descent directions, to construct numerical methods. To solve similar problems in the nonsmooth case, usually, the notion of directional derivative (or, if the function is not directionally differentiable, its some generalization, like

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the Dini and Hadamard upper and lower directional derivatives, the Clarke derivative, the Michel-Penot derivative etc.) is employed. All these derivatives are positively homogeneous functions of direction. For convex and max-type functions the directional derivative is convex (and p.h.) and, by the Minkowski duality, optimality conditions can be stated in geometric terms (see [4,22]). The steepest descent directions can also be derived in this case.

Many attempts were undertaken to find a convex tool in the nonconvex case. Among the most popular ones it is necessary to mention, first of all, the Clarke generalized derivative and the related Clarke subdifferential. However, it turns out that this tool, being very important for some problems, is unable to provide a good approximation or a descent direction in the essentially nonconvex case.

The idea to reduce the problem of minimizing an arbitrary function to a *sequence* of convex problems was implemented by Pschenichnyi [21], who introduced the notions of upper convex and lower concave approximations (u.c.a's and l.c.a.'s). Rubinov [6] proposed to consider exhaustive families of upper convex and lower concave approximations. Later some new tools—upper and lower exhausters and convexificators—closely related to exhaustive families of approximations were introduced. They represent dual objects and allow one to reduce the original optimization problem to a sequence of convex optimization problems.

To explain the idea, note that (see Ref. [6]) if $f : R^n \to R$ is a given directionally differentiable function and $h(g) = f'(x, g)$ is the derivative of the function f at a point x in a direction g and if h is upper semicontinuous in g, then $h(g)$ can be expressed as

$$h(g) = \inf_{C \in E^*} \max_{v \in C} (v, g),$$

and if $h(g) = f'(x, g)$ is lower semicontinuous in g, then $h(g)$ can be written in the form

$$h(g) = \sup_{C \in E_*} \min_{w \in C} (w, g).$$

If h is continuous in g then both above representations are valid. The pair $E = [E^*, E_*]$ of families of convex compact sets is called a biexhauster, E^* being an upper exhauster and E_* - a lower one. The notion of exhauster was introduced in Refs. [8,9,11].

It was shown there that if x^* is a minimizer of f on R^n and an upper exhauster E^* of f at x^* is known then a necessary condition for an unconstrained minimum is

$$0_n \in C \quad \forall C \in E^*. \tag{1}$$

If x^{**} is a maximizer of f on R^n and a lower exhauster E_* of f at x^{**} is known then a necessary condition for an unconstrained maximum takes the form

$$0_n \in C \quad \forall C \in E_*. \tag{2}$$

In Ref. [3] a survey of some results related to these new tools is given. It is shown there, in particular, how to formulate optimality conditions in terms of proper exhausters and to find steepest ascent and descent directions. Since conditions for a minimum are expressed in terms of an upper exhauster and conditions for a maximum are described by means of a lower exhauster, a conversion operator is required to convert upper exhausters into lower ones, and vice versa. One of possible convertors is described in Ref. [3], and a modified convertor is introduced in Ref. [24]. However, recently an attempt to describe optimality conditions for a minimum in terms of a lower (not upper) exhauster and, symmetrically, to describe optimality conditions for a maximum in terms of an upper (instead of lower) exhauster was undertaken in Refs. [10] and [23]). We review these results for the constrained case and provide some illustrative examples. The unconstrained case was discussed in Ref. [5].

In the present paper some further problems related to exhausters are discussed. The paper is organized as follows. In Sect.2 different directional derivatives are described, necessary (as well as sufficient) optimality conditions are formulated in terms of directional derivatives. Necessary and sufficient conditions for a minimum and a maximum in terms of proper exhausters are stated in Sect. 3. Optimality conditions in terms of adjoint exhausters are formulated in Sect. 4. Characterization of Lipschitzivity of a p.h. function is formulated in terms of exhausters (Sect.5). Expressions for the Michel-Penot subdifferential and the Fréchet subdifferential in terms of exhausters are obtained in Sects. 6 and 7. Quasidifferentiability and a necessary and sufficient condition for a function to be quasidifferentiable are discussed in Sects. 8 and 9. Section 10 contains concluding remarks.

2 Optimality conditions via directional derivatives

The notion of directional derivative and its generalizations play an essential role in Nonsmooth Analysis and Nondifferentiable Optimization. In this section we recall the definitions of Dini and Hadamard (upper and lower) directional derivatives and state first-order optimality conditions in terms of these derivatives.

Let $f : X \to R, X \subset R^n$ be an open set. The function f is called Dini directionally differentiable (D-d.d.) at $x \in X$ if for every $g \in R^n$ there exists the finite limit

$$f'_D(x, g) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha}[f(x + \alpha g) - f(x)]. \tag{3}$$

The function f is called Hadamard directionally differentiable (H-d.d.) at $x \in X$ if for every $g \in R^n$ there exists the finite limit

$$f'_H(x, g) = \lim_{[\alpha, g'] \to [+0, g]} \frac{1}{\alpha}[f(x + \alpha g') - f(x)]. \tag{4}$$

Of course, if f is H-d.d. at x, then it is D-d.d. at x and $f'_D(x, g) = f'_H(x, g)$. The inverse statement is not true. The quantity $f'_D(x, g)$ ($f'_H(x, g)$) is called the Dini (Hadamard) derivative of f at x in the direction g.

Let $x \in X$, $g \in R^n$. The quantity

$$f^{\uparrow}_D(x, g) = \limsup_{\alpha \downarrow 0} \frac{1}{\alpha}[f(x + \alpha g) - f(x)] \tag{5}$$

is called the Dini upper derivative of f at x in the direction g.

The quantity

$$f^{\downarrow}_D(x, g) = \liminf_{\alpha \downarrow 0} \frac{1}{\alpha}[f(x + \alpha g) - f(x)] \tag{6}$$

is called the Dini lower derivative of f at x in the direction g.

The quantity

$$f^{\uparrow}_H(x, g) = \limsup_{[\alpha, g'] \to [+0, g]} \frac{1}{\alpha}[f(x + \alpha g') - f(x)] \tag{7}$$

is called the Hadamard upper derivative of f at x in the direction g. The quantity

$$f^{\downarrow}_H(x, g) = \liminf_{[\alpha, g'] \to [+0, g]} \frac{1}{\alpha}[f(x + \alpha g') - f(x)] \tag{8}$$

is called the Hadamard lower derivative of f at x in the direction g. The limits in (5)–(8) always exist (unlike the ones in (3) and (4)) but may be not finite.

If f is D-d.d. then $f_D^{\downarrow}(x, g) = f_D^{\uparrow}(x, g) = f_D'(x, g)$. Analogously, if f is H-d.d. then $f_H^{\downarrow}(x, g) = f_H^{\uparrow}(x, g) = f_H'(x, g)$.

Now consider the problem of minimizing the function f on a set $\Omega \subset X$. Fix $x \in \Omega$ and introduce the following cone

$$\mathcal{K}(x, \Omega) = \{g \in R^n \mid \exists \alpha_k \, : \, \alpha_k \downarrow 0, \; x + \alpha_k g \in \Omega \; \forall k\}.$$

The set

$$\Gamma(x, \Omega) = \{g \in R^n \mid \exists \alpha_k, g_k \, : \, [\alpha_k, g_k] \to [+0, g], \; x + \alpha_k g_k \in \Omega \; \forall k\}$$

is called the *Bouligand cone* to Ω at x. Both these cones are nonempty (they always contain the zero element). The cone $\Gamma(x, \Omega)$ is closed while the cone $\mathcal{K}(x, \Omega)$ is not necessarily closed. If $x \in \Omega$ is an isolated point of Ω then both cones $\mathcal{K}(x, \Omega)$ and $\Gamma(x, \Omega)$ contain only the zero element. We say that $g \in \mathcal{K}^*(x, \Omega)$ if $g \in \mathcal{K}(x, \Omega)$ and

$$\rho(x + \delta g, l \cap \Omega) = o(\delta, x, l),$$

where $l = \{x + \alpha g \mid \alpha \geq 0\}$,

$$\frac{o(\delta, x, l)}{\delta} \underset{\delta \downarrow 0}{\longrightarrow} 0$$

and ρ is the Hausdorff metric. It follows from the definition above that the cone $\mathcal{K}^*(x, \Omega)$ is not empty (it contains at least zero) and that $\mathcal{K}^*(x, \Omega) \subset \mathcal{K}(x, \Omega)$.

We say that a closed cone Γ is a first-order uniform approximation (or a first-order uniform conical approximation) of a set Ω near a point $x \in \Omega$ if

$$\rho(\Omega \cap \mathcal{B}_\delta(x), (x + \Gamma) \cap \mathcal{B}_\delta(x)) = o_x(\delta),$$

where

$$\frac{o_x(\delta)}{\delta} \underset{\delta \downarrow 0}{\longrightarrow} 0.$$

Here $\mathcal{B}_\delta$ is a closed ball of radius $\delta \geq 0$ centered at $x \in R^n$. The notion of first-order approximation was introduced in Ref. [6].

The following two Lemmas provide the first-order constrained optimality conditions in terms of directional derivatives and approximating cones.

Lemma 1 *Let f be locally Lipschitz around a point $x^* \in \Omega$. For the point x^* to be a local or global minimizer of the function f on the set Ω it is necessary that*

$$f_D^{\downarrow}(x^*, g) \geq 0 \quad \forall g \in \mathcal{K}^*(x^*, \Omega). \tag{9}$$

If the cone $\Gamma(x^*, \Omega)$ is a first order uniform approximation of Ω near the point $x^* \in \Omega$ then for the point x^* to be a local or global minimizer of the function f on the set Ω it is necessary that

$$f_H^{\downarrow}(x^*, g) \geq 0 \quad \forall g \in \Gamma(x^*, \Omega). \tag{10}$$

The condition

$$f_H^{\downarrow}(x^*, g) > 0 \quad \forall g \in \Gamma(x^*, \Omega), \quad g \neq 0_n \tag{11}$$

is sufficient for x^* to be a strict local minimizer of f on Ω.

Lemma 2 *Let f be locally Lipschitz around a point $x^{**} \in \Omega$. For the point x^{**} to be a local or global maximizer of the function f on the set Ω it is necessary that*

$$f_D^{\uparrow}(x^{**}, g) \geq 0 \quad \forall g \in \mathcal{K}^*(x^{**}, \Omega). \tag{12}$$

If the cone $\Gamma(x^{**}, \Omega)$ is a first order uniform approximation of Ω near the point $x^{**} \in \Omega$ then for the point x^{**} to be a local or global maximizer of the function f on the set Ω it is necessary that

$$f_H^{\uparrow}(x^{**}, g) \geq 0 \quad \forall g \in \Gamma(x^{**}, \Omega). \tag{13}$$

The condition

$$f_H^{\uparrow}(x^{**}, g) > 0 \quad \forall g \in \Gamma(x^{**}, \Omega), \quad g \neq 0_n \tag{14}$$

is sufficient for x^{**} to be a strict local maximizer of f on Ω.

A point $x^* \in X$ satisfying (9) (10) is called a Dini (Hadamard) inf-stationary point of f.

Note that all the functions defined by (3)–(8) are positively homogeneous (p.h.) of the first degree as functions of g, hence, exhausters can be employed to study them.

3 Optimality conditions in terms of proper exhausters

It has already been observed (see Lemmas 1–2) that the first-order optimality conditions can be expressed in terms of corresponding directional derivatives. Here these conditions are reformulated in terms of proper exhausters.

An upper exhauster $E^*(h)$ will be referred to as *a proper exhauster* for the minimization problem, while a lower exhauster $E_*(h)$ is called *a proper exhauster* for the maximization problem.

Let $C \subset R^n$, by $K(C)$ we denote the cone conjugate to C:

$$K(C) = \{w \in R^n \mid (w, v) \geq 0 \; \forall v \in C\},$$

and put $\kappa(C) = \{w \in R^n \mid (w, v) > 0 \; \forall v \in C\}$. By $N(C)$ we denote the normal cone of C:

$$N(C) = \{w \in R^n \mid (w, v) \leq 0 \; \forall v \in C\},$$

and put $\eta(C) = \{w \in R^n \mid (w, v) < 0 \; \forall v \in C\}$.

Note that

$$\kappa(C) \subset \text{int}\, K(C), \;\; \eta(C) \subset \text{int}\, N(C),$$

but not necessarily $\kappa(C) = \text{int}\, K(C)$ or $\eta(C) = \text{int}\, N(C)$. The sets $\kappa(C)$ and $\eta(C)$ can even be empty while $K(C)$ and $N(C)$ are always nonempty (since $0_n \in K(C)$ and $0_n \in N(C)$).

Let Γ be a cone with the apex 0_n and let

$$\Gamma = \cup \{A \mid A \in \mathcal{A}\}, \tag{15}$$

where $\mathcal{A}$ is a family of convex cones with the apex 0_n. Every cone can be represented in the form (5) (for example, it is possible to take as $\mathcal{A}$ the family of all rays in Γ).

The following four lemmas (see Refs. [3] and [10]) demonstrate the constrained optimality conditions in terms of proper exhausters.

Lemma 3 *Let $h : R^n \to R$ be a p.h. function and assume that there exists an upper exhauster $E^*(h)$ of h. Then the following statements are equivalent:*

$$
\begin{aligned}
&(1)\quad h(g) \geq 0 \quad \forall g \in \Gamma;\\
&(2)\quad C \cap K(A) \neq \emptyset \quad \forall C \in E^*(h),\ \forall A \in \mathcal{A};\\
&(3)\quad 0_n \in [C - K(A)] \quad \forall C \in E^*(h),\ \forall A \in \mathcal{A};\\
&(4)\quad 0_n \in L^*(h, \Gamma) := \cap\{[C - K(A)] \mid C \in E^*(h),\ A \in \mathcal{A}\}.\\
&(5)\quad \Gamma \subset R^n \setminus \bigcup_{C \in E^*(h)} \eta(C).
\end{aligned}
\tag{16}
$$

Lemma 4 *Let $h : R^n \to R$ be a p.h. function and assume that there exists an upper exhauster $E^*(h)$ of h. Then the following statements are equivalent:*

$$(1)\quad h(g) > 0 \quad \forall g \in \Gamma,\ g \neq 0_n;$$

(2) *There exists a $\delta > 0$ such that, for every $C \in E^*(h)$ and $A \in \mathcal{A}$, a point $V_{CA} \in R^n$ exists such that* $\mathcal{B}_\delta(V_{CA}) \subset [C \cap K(A)]$;

(3) *There exists a $\delta > 0$ such that $\mathcal{B}_\delta \subset [C - K(A)]$ $\forall C \in E^*(h),\ \forall A \in \mathcal{A}$;*

$$(4)\quad 0_n \in \operatorname{int} L^*(h, \Gamma).$$

Lemma 5 *Let $h : R^n \to R$ be a p.h. function and assume that there exists a lower exhauster $E_*(h)$ of h. Then the following statements are equivalent:*

$$
\begin{aligned}
&(1)\quad h(g) \leq 0 \quad \forall g \in \Gamma;\\
&(2)\quad (-C) \cap K(A) \neq \emptyset \quad \forall C \in E_*(h), \forall A \in \mathcal{A};\\
&(3)\quad 0_n \in [C + K(A)] \quad \forall C \in E_*(h),\ \forall A \in \mathcal{A};\\
&(4)\quad 0_n \in L_*(h, \Gamma) := \cap\{[C + K(A)] \mid C \in E_*(h),\ A \in \mathcal{A}\}.\\
&(5)\quad \Gamma \subset R^n \setminus \bigcup_{C \in E_*(h)} \kappa(C).
\end{aligned}
\tag{17}
$$

Lemma 6 *Let $h : R^n \to R$ be a p.h. function and assume that there exists a lower exhauster $E_*(h)$ of h. Then the following statements are equivalent:*

$$(1)\quad h(g) < 0 \quad \forall g \in \Gamma,\ g \neq 0_n;$$

(2) *There exists a $\delta > 0$ such that, for every $C \in E_*(h)$ and*

$A \in \mathcal{A}$, a point $V_{CA} \in R^n$ exists such that

$$\mathcal{B}_\delta(V_{CA}) \subset [-C \cap K(A)];$$

(3) *There exists a* $\delta > 0$ *such that*

$$\mathcal{B}_\delta \subset [C + K(A)] \ \ \forall C \in E_*(h), \ \forall A \in \mathcal{A};$$

(4) $0_n \in \operatorname{int} L_*(h, \Gamma)$.

Remark 1 Note that to check the condition (5) in Lemmas 3 and 5 one does not need to find a decomposition of Γ, however it is necessary to construct the sets $\kappa(C)$ for all $C \in E_*(h)$ for a lower exhauster or $\eta(C)$ for an upper one.

4 Optimality conditions in terms of adjoint exhausters

Let Γ be a cone in R^n. Then the following statements are valid.

Lemma 7 *Let* $h : R^n \to R$ *be a p.h. function and assume that there exists an upper exhauster* $E^*(h)$ *of* h. *Then the following statements are equivalent.*

(1) $h(g) < 0 \ \ \forall g \in \Gamma \setminus \{0_n\}$;

(2) $\{\Gamma \setminus \{0_n\}\} \subset \bigcup_{C \in E^*(h)} \eta(C)$.

Lemma 8 *Let* $h : R^n \to R$ *be a p.h. function and assume that there exists an upper exhauster* $E^*(h)$ *of* h. *If*

$$\Gamma \subset \bigcup_{C \in E_*(h)} N(C),$$

then

$$h(g) \leq 0 \ \ \forall g \in \Gamma.$$

Moreover, if the function $h(g)$ *can be represented in the form*

$$h(g) = \min_{C \in E^*} \max_{v \in C} (v, g), \tag{18}$$

where the family E^* *is an upper exhauster of* h, *then the conditions* (1) *and* (2) *are equivalent.*

Lemma 9 *Let* $h : R^n \to R$ *be a p.h. function and assume that there exists a lower exhauster* $E_*(h)$ *of* h. *Then the following statements are equivalent:*

(1) $h(g) > 0 \ \ \forall g \in \Gamma \setminus \{0_n\}$;

(2) $\Gamma \setminus \{0_n\} \subset \bigcup_{C \in E_*(h)} \kappa(C)$.

Lemma 10 *Let $h : R^n \to R$ be a p.h. function and assume that there exists a lower exhauster $E_*(h)$ of h. If*

$$(1) \quad \Gamma \subset \bigcup_{C \in E_*(h)} K(C), \tag{19}$$

then

$$(2) \quad h(g) \geq 0 \quad \forall g \in \Gamma.$$

Moreover, if the function $h(g)$ can be represented in the form

$$h(g) = \max_{C \in E_*} \min_{v \in C} (v, g), \tag{20}$$

where E_* is a lower exhauster of h, then the conditions (1) *and* (2) *are equivalent.*

Remark 2 It turns out that via proper exhausters it is possible to construct steepest ascent and descent directions (see, e.g. [3]). Making use of adjoint exhausters, one is able to find some ascent and descent directions. The problem of constructing steepest ascent and descent directions by means of adjoint exhausters is still open.

5 Necessary and sufficient conditions for Lipschitzivity in terms of exhausters

Theorem 1 *Let $h : R^n \to R$ be a p.h. function. For the function h to be Lipschitz it is necessary and sufficient that there exist a totally bounded upper exhauster and a totally bounded lower exhauster.*

Proof *Necessity.* In Ref. [1] M.Castellani proved that if h is Lipschitz then h can be written in the forms

$$h(g) = \min_{C \in E^*} \max_{v \in C} (v, g) \quad \forall g \in R^n \tag{21}$$

and

$$h(g) = \max_{C \in E_*} \min_{w \in C} (w, g) \quad \forall g \in R^n, \tag{22}$$

where the families of sets E^* and E_* are totally bounded. Remind that a family of sets E is totally bounded if there exists a ball B in R^n such that $C \subset B \ \forall C \in E$.

The relation (21) implies that E^* is an upper exhauster of h and E^* is totally bounded. The relation (22) implies that E_* is a lower exhauster of h and E_* is totally bounded.

Sufficiency. The Lipschitzivity of h is obvious from the representations (21), (22) and the total boundedness of the families of sets E^* and E_*.

6 The Michel-Penot subdifferential in terms of exhausters

Assume that h is p.h. and Lipschitz. As was indicated above, h can be represented in the forms (21) and (22). First consider the representation (21). Put

$$Q(g) = \{C \in E^* \mid h(g) = \max_{v \in C} (v, g)\}, \tag{23}$$

$$h_C(g) = \max_{v \in C} (v, g), \tag{24}$$

$$V_g(C) = \{w \in C \mid (w, g) = h_C(g) = \max_{v \in C}(v, g)\}. \tag{25}$$

Then

$$h(g) = \min_{C \in E^*} \max_{v \in C}(v, g) = \min_{C \in Q(g)} \max_{w \in V_g(C)} (w, g) = \min_{w \in E_g^*} (w, g) \quad \forall g \in R^n, \tag{26}$$

where

$$E_g^* = \text{cl co}\{V_g(C) \mid C \in Q(g)\}. \tag{27}$$

Now let us consider the polyhedral case: Assume that the family E^* contains a finite number of sets and every set $C \in E^*$ is a polyhedron. Then the function h is directionally differentiable at every point $g \in R^*$ and

$$h'(g, q) = \min_{C \in Q(g)} \max_{w \in V_g(C)} (w, q). \tag{28}$$

Furthermore, for almost all g the sets $Q(g)$ and $V_g(C)$ are singletons, and, hence, the set E_g^* is also a singleton:

$$E_g^* = \{w_g\}.$$

It means that the function h is almost everywhere differentiable(the differentiability follows also from the Lipschitzivity of h) and therefore for almost every g there exists the gradient $h'(g)$ of h and

$$h'(g) = w_g. \tag{29}$$

By $T(h)$ we denote the set of points of differentiability of h. The set $T(h)$ is a set of full measure and obviously $0_n \notin T(h)$.

It is well known (see Ref. [2]) that the set

$$\partial_{Cl} h(0_n) = \text{cl co}\{w_g \mid g \in T(h)\} \tag{30}$$

is the Clarke subdifferential of h at 0_n. Now one is able to express the Clarke subdifferential of h at 0_n constructively via points w_g.

If a function $f : R^n \to R$ is Lipschitz and directionally differentiable at $x \in R^n$ and $h(g)$ is its directional derivative at the point x then $\partial_{Cl} h(0_n)$ is the Michel-Penot subdifferential (see Ref. [15]) of the function f at the point x (called also "the small subdifferential" [13]):

$$\partial_{MP} f(x) = \partial_{Cl} h(0_n) = \text{cl co}\{w_g \mid g \in T(h)\} \subset \partial_{Cl} f(x). \tag{31}$$

Hence, the Michel-Penot subdifferential of f at x can be constructed by means of the upper exhauster of the directional derivative $h(g) = f'(x, g)$. In some cases (see Ref. [7]) the Michel-Penot subdifferential coincides with the Clarke subdifferential.

Remark 3 In the nonpolyhedral case similar results are valid under some additional condition.

Remark 4 Analogous results (with proper alterations) can be formulated if one uses the representation (22) instead of (21). In this case the lower exhauster E_* will be employed.

7 The Fréchet subdifferential in terms of upper exhausters

For a function $\varphi : R^n \to R$ the Fréchet subdifferential of φ at $\bar{x}$ can be defined as follows (see Refs. [14,17]):

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ v \in R^n \;\middle|\; \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - (v, x - \bar{x})}{||x - \bar{x}||} \geq 0 \right\}.$$

Let $h : R^n \to R$ be a p.h. and upper semicontinuous function. It is not difficult to observe that in this case the Fréchet subdifferential of the function h at zero is

$$\widehat{\partial}h(0_n) = \left\{ v \in R^n \mid h(x) - (v, x) \geq 0 \ \forall x \in R^n \right\}. \tag{32}$$

Theorem 2 *Let E^* be an upper exhauster of a p.h. function $h : R^n \to R$. Then*

$$\bigcap_{C \in E^*} C = \widehat{\partial}h, \tag{33}$$

where $\widehat{\partial}h$ is the Fréchet subdifferential of h at 0_n.

Proof Denote $E = \bigcap_{C \in E^*} C$. Take an arbitrary $v_0 \in E$. It follows from the definition of an upper exhauster that

$$(v_0, x) \leq h(x) \ \forall x \in R^n.$$

Hence, $v_0 \in \widehat{\partial}h$. Due to the arbitrariness of v_0 one gets

$$E \subseteq \widehat{\partial}h. \tag{34}$$

Consider now any $v_0 \in \widehat{\partial}h$. The relation (34) yields

$$h(x) \geq (v_0, x) \quad \forall x \in R^n. \tag{35}$$

Suppose now that there exists $C_0 \in E^*$ such that $v_0 \notin C_0$. Then by the separation theorem (see, e.g., [22]) there exists $x_0 \in R^n$ such that

$$(x_0, v_0) > \max_{v \in C_0} (x_0, v) \geq h(x_0),$$

which contradicts (35). Hence, $v_0 \in C$ for every $C \in E^*$ and due to the arbitrariness of v_0 one concludes

$$\widehat{\partial}h \subseteq E. \tag{36}$$

Now (34) and (36) yield (33).

Corollary 1 *Let E_1^* and E_2^* be two upper exhausters of the same p.h. function $h : R^n \to R$. Then*

$$\bigcap_{C \in E_1^*} C = \bigcap_{C \in E_2^*} C.$$

Proof follows from the uniqueness of the Fréchet subdifferential and Theorem 2.

The following example illustrates the results obtained in Theorem 2.

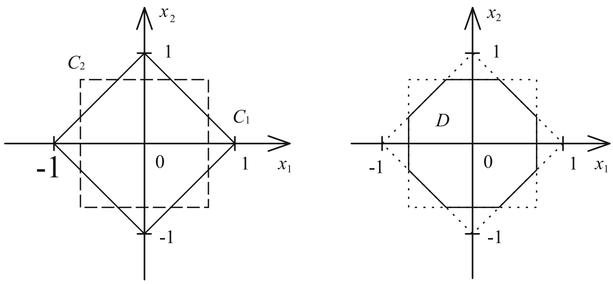

Fig. 1 Example 1: an upper exhauster and the Fréchet subdifferential

Example 1 Consider the function $h : R^2 \to R$,

$$h(x_1, x_2) = \min\left\{\frac{1}{\sqrt{2}}(|x_1| + |x_2|), \max\{|x_1|, |x_2|\}\right\}.$$

It is not difficult to see that $E^* = \{C_1, C_2\}$, where

$$C_1 = \text{co}\,\{(0, 1), (0, -1), (-1, 0), (1, 0)\};$$

$$C_2 = \text{co}\,\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\},$$

is an upper exhauster of h (see Fig. 1). Now one can easily obtain the Fréchet subdifferential of h at zero by calculating the intersection of C_1 and C_2. We do not need to use a more complicated expression (32) (in Fig. 1 the set $\widehat{\partial} h(0_n) = D$ is indicated by the solid line).

Remark 5 Let a p.h. function $h(g)$ be the directional derivative of a function $f : R^n \to R$ at a point $x : h(g) = f'(x, g)$. It follows from Theorem 2 that the condition (2) in Lemma 3 is equivalent to the condition

$$0 \in \widehat{\partial} h(0_n), \tag{37}$$

which is in turn the well-known necessary condition for the unconstrained minima of f (see, for example, [16]). While necessary conditions for a minimum in terms of exhausters coincide with the necessary conditions via the Fréchet subdifferential, they are constructive in the sense that one is able to find steepest descent directions via upper exhauster, but not via Fréchet subdifferential (which in many cases is even the empty set), when the necessary conditions for a minimum are not satisfied.

To illustrate the above remark, consider the following example.

Example 2 Let $h(x) = \min\{\max\{y, -2x - y\}, \max\{y, 2x - y\}\}$. Its upper exhauster is $E^* = \{C_1, C_2\}$, where $C_1 = \{(0, 1), (-2, -1)\}$, $C_2 = \{(0, 1), (2, -1)\}$ (see Fig. 2). It is easy to see that $\widehat{\partial} h(0_n) = C_1 \cap C_2 = \{(0, 1)\}$, and the necessary condition for the minima is not satisfied. Hence, we can find directions of steepest descent using the techniques described in Ref. [3], that is, for every $C \in E^*$ find $d_C = \min_{v \in C} ||v|| = ||v_C|| > 0$ and take $g_C = -v_C/||v_C||$. Then if $C^* \in E^*$ is such that $g(C^*) = \sup_{C \in E^*} d(C)$, the direction g_{C^*} is a steepest descent direction. In our case there exist two steepest descent directions: $g_1 = (1/\sqrt{2}, -1/\sqrt{2})$ and $g_2 = (-1/\sqrt{2}, -1/\sqrt{2})$. Note that we cannot obtain these direction using the Fréchet subdifferential. Indeed, since $\widehat{\partial} h(0_n) = \{(0, 1)\}$, then $(0, 1)$ is the

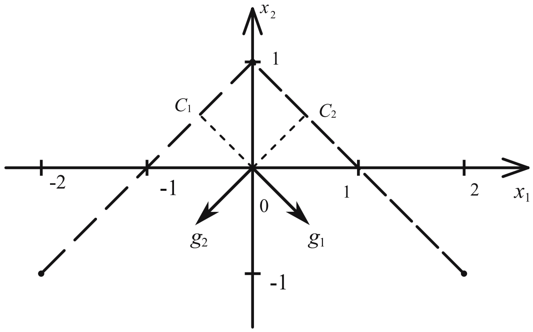

Fig. 2 Example 2

nearest point of $\widehat{\partial}h(0_n)$ to the origin. However, the direction $g = (0, -1)$ is not a steepest descent direction.

Note that symmetrically to the Fréchet subdifferential it is also possible to define the corresponding Fréchet upper subdifferential:

$$\widehat{\partial}^{+}\varphi(\bar{x}) := \left\{ v \in R^n \;\middle|\; \limsup_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - (v, x - \bar{x})}{||x - \bar{x}||} \leq 0 \right\},$$

which for a positively homogeneous function $h : R^n \to R$ at zero takes the following form:

$$\widehat{\partial}^{+}h(0_n) = \left\{ v \in R^n \mid h(x) - (v, x) \leq 0 \quad \forall x \in R^n \right\}.$$

Then the similar results can be stated about the upper Fréchet subdifferential and a lower exhauster.

Theorem 3 *Let E_* be a lower exhauster of a p.h. function $h : R^n \to R$. Then*

$$\bigcap_{C \in E_*} C = \widehat{\partial}^{+}h(0_n), \tag{38}$$

where $\widehat{\partial}h$ is the Fréchet subdifferential of h at 0_n.

Corollary 2 *Let E_*^1 and E_*^2 be two lower exhausters of the same p.h. function $h : R^n \to R$. Then*

$$\bigcap_{C \in E_*^1} C = \bigcap_{C \in E_*^2} C.$$

8 Quasidifferentiable functions

It follows from the aforesaid that the problems of verifying necessary and/or sufficient optimality conditions and computing descent, steepest descent, ascent, and steepest ascent directions of some (generally speaking nondifferentiable) function are reduced to solving some geometric problems and finding nearest points to some convex sets. To employ the above results it is required to be able to construct the corresponding exhausters. For some classes of functions such tools are available. For example, if f is a quasidifferentiable function then [7] its directional derivative at a point x is represented as

$$f'(x,g) = \max_{v \in \underline{\partial} f(x)} (v,g) + \min_{w \in \overline{\partial} f(x)} (w,g), \tag{39}$$

where $\underline{\partial} f(x), \overline{\partial} f(x) \subset R^n$ are compact convex sets. It is clear from (39) that

$$f'(x,g) = \min_{w \in \overline{\partial} f(x)} \max_{v \in w + \underline{\partial} f(x)} (v,g) = \max_{v \in \underline{\partial} f(x)} \min_{w \in v + \overline{\partial} f(x)} (w,g).$$

Hence, for the function $h(g) = f'(x,g)$ we get

$$E^*(h) = \{C = w + \underline{\partial} f(x) \mid w \in \overline{\partial} f(x)\}, \tag{40}$$

$$E_*(h) = \{C = v + \overline{\partial} f(x) \mid v \in \underline{\partial} f(x)\}. \tag{41}$$

Condition (1) and relation (40) (applied to the function $h(g) = f'(x^*, g)$) provide the following necessary condition for a minimum of a quasidifferentiable function: $0_n \in w + \underline{\partial} f(x^*) \quad \forall w \in \overline{\partial} f(x^*)$, which is equivalent to (see [7,20]) $-\overline{\partial} f(x^*) \subset \underline{\partial} f(x^*)$.

Condition (2) and relation (41) (applied to the function $h(g) = f'(x^{**}, g)$) provide the following necessary condition for a maximum of a quasidifferentiable function: $0_n \in v + \overline{\partial} f(x^{**}) \quad \forall v \in \underline{\partial} f(x^{**})$, which is equivalent to (see [7,20]) $-\underline{\partial} f(x^{**}) \subset \overline{\partial} f(x^{**})$.

9 A necessary and sufficient condition for quasidifferentiability in terms of exhausters

Let $E \in 2^{R^n}$ be a family of convex sets in R^n. We say that the family E is of *translation type* if there exists a convex set $C_0 \subset R^n$ such that for every $C \in E$ there is a point $w_C \in R^n$ such that $C = C_0 + w_C$.

A p.h. function $h(g)$ will be referred to as *quasidifferentiable* if there exist convex compact sets $A \subset R^n$ and $B \subset R^n$ such that

$$h(g) = \max_{v \in A}(v,g) + \min_{w \in B}(w,g). \tag{42}$$

It follows from Sect. 8 that the directional derivative of a quasidifferentiable function (as a function of direction) is p.h. and quasidifferentiable and both the upper exhauster and the lower one are of translation type and totally bounded. The following characterization of a quasidifferentiable p.h. function is valid.

Theorem 4 *Let $h : R^n \to R$ be a p.h. function. For the function h to be quasidifferentiable it is necessary and sufficient that there exist a totally bounded translation-type upper exhauster and a totally bounded translation-type lower exhauster.*

Proof Sufficiency follows from the arguments in Sect. 8.

Necessity. We consider only the case of the upper exhauster. Let E^* be an upper exhauster of h and there exists $C_0 \subset R^n$ such that for every $C \in E^*$ one can find $w_C \in R^n$ such that $C = C_0 + w_C$. Then

$$\begin{aligned} h(g) &= \min_{C \in E^*} \max_{v \in C}(v,g) = \min_{C \in E^*} \max_{v \in [C_0 + w_C]} (v,g) = \min_{C \in E^*} \{\max_{v \in C_0}(v,g) + (w_C, g)\} \\ &= \max_{v \in C_0}(v,g) + \min_{C \in E^*}(w_C, g) \quad \forall g \in R^n. \end{aligned} \tag{43}$$

It follows from (43) that

$$h(g) = \max_{v \in A}(v, g) + \min_{w \in B}(w, g) \quad \forall g \in R^n \tag{44}$$

where $A = C_0$, $B = \text{co}\{w \in R^n \mid w = w_C,\ C \in E^*\}$. The representation (44) implies the quasidifferentiability of h.

Remark 6 It follows from Theorem 1 that a quasidifferentiable p.h. function is Lipschitz. An example of a p.h. function which is not Lipschitz (and, hence, is not quasidifferentiable) was given by Glover et al. [12].

10 Concluding remarks

Remark 7 In the current paper we have reviewed some properties of exhausters, including necessary and sufficient conditions for an extremum and the ways to find steepest ascent and descent directions. It can easily be seen from the above discussion, that almost every function of practical interest can be studied by means of upper and lower exhausters. Moreover, the problems of checking extremal conditions and finding steepest directions are reduced to a sequence of convex problems, which, in turn, can be easily solved by existing (convex) tools. The problem of constructing exhausters of an arbitrary function remains important. Some elements of the calculus of exhausters were introduced in [3]. Their application to constrained optimization problems is discussed in [25].

Remark 8 Necessary and sufficient conditions for Lipschitzivity and quasidifferentiability in terms of exhausters were introduced.

Remark 9 It was shown in Sects. 6 and 7 that exhausters are closely related to other nonsmooth tools, such as Michel-Penot, Clarke and Fréchet subdifferentials. Note that the discovered relations are all in the form of equalities.

Remark 10 Since exhausters are not uniquely defined, the problem of minimality arises (like the similar problem with quasidifferentials [18,19]). The problems of existence, uniqueness and construction of minimal exhausters are still open.

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